Homotopy Perturbation Method for Solving Partial Differential Equations

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We apply a relatively new technique which is called the homotopy perturbation method (HPM) for solving linear and nonlinear partial differential equations. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. Several examples are given to verify the reliability and efficiency of the method. The fact that the HPM solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

Key words: Homotopy Perturbation Method; Partial Differential Equations; Helmholtz Equations; Fisher's Equations; Initial Boundary Value Problems; Boussinesq Equations.

1. Introduction

In the last two decades with the rapid development of nonlinear science, there has appeared ever-increasing interest of physicists and engineers in the analytical techniques for nonlinear problems. It is well known, that perturbation methods provide the most versatile tools available in nonlinear analysis of engineering problems (see [1 – 19] and the references therein). The perturbation methods, like other nonlinear analytical techniques, have their own limitations. At first, almost all perturbation methods are based on the assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to the ideal results, but an unsuitable choice may create serious problems. Furthermore, the approximate solutions solved by perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption. These facts have motivated to suggest alternate techniques, such as variational iteration [8, 15 – 18, 20 – 32], decomposition [33 – 39], exp-function [40, 41], variation of parameters [42] and iterative [43, 44]. In order to overcome these drawbacks, combining the standard homotopy and perturbation method, which is called the homotopy perturbation, modifies the homotopy method.

Many problems in natural and engineering sciences are modeled by partial differential equations (PDEs). These equations arise in a number of scientific models such as the propagation of shallow water waves, long wave and chemical reaction-diffusion models (see [14, 15, 32 – 41, 45 – 65] and the references therein). A substantial amount of work has been invested for solving such models. Several techniques including the method of characteristic, Riemann invariants, combination of waveform relaxation and multi-grid, periodic multi-grid wave form, variational iteration, homotopy perturbation and Adomian's decomposition [14, 15, 32 – 41, 45 – 65] have been used for the solutions of such problems. Most of these techniques encounter the inbuilt deficiencies and involve huge computational work. He [3 – 8] developed the homotopy perturbation method for solving linear, nonlinear, initial and boundary value problems by merging two techniques, the standard homotopy and the perturbation technique. The homotopy perturbation method was formulated by taking the full advantage of the standard homotopy and perturbation methods and has been applied to a wide class of functional equations (see [1 – 19] and the references therein). The basic motivation of the present paper is the implementation of this reliable technique for solving PDEs. In particular the proposed homotopy perturbation method (HPM) is tested on Helmholtz, Fisher's, Boussinesq, singular fourth-order partial differential equations, systems of partial differential equations and higher-dimensional initial boundary value problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions and is free from round off errors. The HPM gives the solution in the form of a convergent series with easily computable components. Unlike the method of separation of variables which requires both initial and boundary conditions, the HPM gives the solution by using the initial conditions only. The fact that the proposed HPM solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

2. The Homotopy Perturbation Method

To explain the HPM, we consider a general equation of the type

$$L(u) = 0, \tag{1}$$

where L is any integral or differential operator. We define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)F(u) + pL(u), \tag{2}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that for

$$H(u,p) = 0, \tag{3}$$

we have

$$H(u,0) = F(u), \quad H(u,1) = L(u).$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point H $(v_0,0)$ to a solution function H $(f,1)$. The embedding parameter monotonically increases from zero to unity as the trivial problem $F(u) = 0$ continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [1 – 19]. The HPM uses the homotopy parameter p as an expanding parameter [3 – 8] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots. \tag{4}$$

If $p \to 1$, then (4) corresponds to (2) and becomes the approximate solution of the form

$$f = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \ldots. \tag{5}$$

It is well known that series (5) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$. For more details about the convergence of the HPM (see [1 – 19] and the references therein). The comparisons of equal powers of p give solutions of various orders. In sum, according to [1,2], He's HPM considers the solution $u(x)$ of the homotopy equation in a series of p as

$$u(x) = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2u_2 + \ldots,$$

and the method considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + pH_1 + p^2H_2 + \ldots,$$

where H_n are the so-called He's polynomials [1,2], which can be calculated by using the formula

$$H_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0},$$
$$n = 0,1,2,\ldots.$$

3. Numerical Applications

In this section, we apply the HPM for solving PDEs. In particular the proposed HPM is tested on Helmholtz, Fisher's, Boussinesq, singular fourth-order partial differential equations, systems of partial differential equations and higher-dimensional initial boundary value problems. Numerical results are very encouraging.

3.1. Example 1

Consider the Helmholtz equation [44]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} - u(x,y) = 0$$

with the initial conditions

$$u(0,y) = y, \quad u_x(0,y) = y + \cosh y.$$

x	y	Exact solution	HPM	ITM	Absolute error
−1	−1	−1.9109600760	−1.9097472990	−1.9097472990	0.0012127770
−0.8	−0.8	−1.4294111280	−1.491500900	−1.491500900	0.0002610380
−0.6	−0.6	−1.0405661130	−1.0405303310	−1.0405303310	0.0000357820
−0.4	−0.4	−0.7005569670	−0.7005548150	−0.7005548150	0.0000021520
−0.2	−0.2	−0.367755010	−0.3677594840	−0.3677594840	0.0000000170
0.0	−0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.2	0.4482939020	0.4482938840	0.4482938840	0.0000000180
0.4	0.4	1.0291588280	1.0291564150	1.0291564150	0.0000024130
0.6	0.6	1.8045504110	1.8045079310	1.8045079310	0.0000424800
0.8	0.8	2.850380700	2.8500524900	2.8500524900	0.0003282100
1.0	1.0	4.2613624630	4.2597472990	4.2597472990	0.0016151640

Table 1. Error estimates.

Applying the convex homotopy method

$$\frac{\partial^2 u_0}{\partial x^2}+p\frac{\partial^2 u_1}{\partial x^2}+p^2\frac{\partial^2 u_2}{\partial x^2}+\ldots = p\left(u_0+pu_1+p^2u_2+\ldots\right) -p\left(\frac{\partial^2 u_0}{\partial y^2}+p\frac{\partial^2 u_1}{\partial y^2}+p^2\frac{\partial^2 u_2}{\partial y^2}+\ldots\right),$$

and comparing the coefficients of equal powers of p

$$p^{(0)}: u_0(x,y)=y(1+x)+x\cosh y,$$
$$p^{(1)}: u_1(x,y)=\frac{1}{2!}x^2y+\frac{1}{3!}x^3y,$$
$$p^{(2)}: u_2(x,y)=\frac{1}{4!}x^4y+\frac{1}{5!}x^5y,$$
$$p^{(3)}: u_3(x,y)=\frac{1}{6!}x^6y+\frac{1}{7!}x^7y,$$
$$p^{(4)}: u_4(x,y)=\frac{1}{8!}x^8y+\frac{1}{9!}x^9y,$$
$$\vdots$$

gives the solution as

$$u(x,y)=y\exp(x)+x\cosh(y).$$

Table 1 exhibits the approximate solution obtained by using the HPM and ITM. It is clear that the obtained results are in high agreement with those obtained using the exact solutions. Higher accuracy can be obtained by using more terms.

3.2. Example 2

Consider the Helmholtz equation [44]

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}+8u(x,y)=0$$

with the initial conditions

$$u(0,y)=\sin(2y),\quad u_x(0,y)=0.$$

Applying the convex homotopy method

$$\frac{\partial^2 u_0}{\partial x^2}+p\frac{\partial^2 u_1}{\partial x^2}+p^2\frac{\partial^2 u_2}{\partial x^2}+\ldots = -p\left[8\left(u_0+pu_1+p^2u_2+\ldots\right) +\left(\frac{\partial^2 u_0}{\partial y^2}+p\frac{\partial^2 u_1}{\partial y^2}+p^2\frac{\partial^2 u_2}{\partial y^2}+\ldots\right)\right],$$

and comparing the coefficients of equal powers of p

$$p^{(0)}: u_0(x,y)=\sin(2y),$$
$$p^{(1)}: u_1(x,y)=-2x^2\sin(2y),$$
$$p^{(2)}: u_2(x,y)=\frac{8}{3}x^4\sin(2y),$$
$$p^{(3)}: u_3(x,y)=-\frac{4}{45}x^6\sin(2y),$$
$$\vdots$$

gives the series solution as

$$u(x,y)=\sin(2y)\left(1-2x^2+\frac{2}{3}x^4-\frac{4}{45}x^6+\ldots\right),$$

and, in the closed form, as

$$u(x,y)=\cos(2x)\sin(2y).$$

Table 2 exhibits the approximate solution obtained by using the HPM and ITM [52]. It is clear that the obtained results are in high agreement with those obtained using the exact solutions. Higher accuracy can be obtained by using more terms.

Table 2. Error estimates.

x	y	Exact solution	HPM	ITM	Absolute error
0.0	0.0	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.1	0.194709171	0.194532593	0.194532593	0.0001765771
0.2	0.2	0.358678045	0.358678045	0.358678045	0.000000000
0.3	0.3	0.466019543	0.466019543	0.466019543	0.000000000
0.4	0.4	0.499786801	0.499786823	0.499786823	0.000000022
0.5	0.5	0.454648713	0.454648957	0.454648957	0.000000244
0.6	0.6	0.337731590	0.337733230	0.337733230	0.000001640
0.7	0.7	0.167494075	0.167501984	0.167501984	0.000007909
0.8	0.8	−0.029187071	−0.029157491	−0.029157491	0.000029580
0.9	0.9	−0.221260221	−0.221170295	−0.221170295	0.000089926
1.0	1.0	−0.378401247	−0.378173063	−0.378173063	0.000228184

3.3. Example 3

Consider the Fisher's equation of the form

$$u_t(x,t) - u_{xx}(x,t) - u(x,t)(1-u(x,t)) = 0,$$

subject to the initial conditions

$$u(x,0) = \beta.$$

This Fisher's problem can be formulated as the integral equation

$$u(x,t) = \beta + \int_0^t (u_{xx}(x,t) + u(x,t))(1-u(x,t))\mathrm{d}t.$$

Applying the convex homotopy method

$$\begin{aligned} &u_0 + pu_1 + p^2u_2 + \ldots = \\ &\beta + p\int_0^t \left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \ldots\right) \\ &+(u_0 + pu_1 + p^2u_2 + \ldots) \\ &\quad \cdot (1-(u_0 + pu_1 + p^2u_2 + \ldots))\mathrm{d}t, \end{aligned}$$

and comparing the coefficients of equal powers of p

$$p^{(0)} : u_0(x,y) = \beta,$$

$$p^{(1)} : u_1(x,y) = \beta(1-\beta)t,$$

$$p^{(2)} : u_2(x,y) = \frac{t^2}{2!}\beta(1-3\beta+2\beta^2) - \frac{t^3}{3}\beta^2(-1+\beta)^2,$$

$$\begin{aligned} p^{(3)} : u_3(x,y) = &\frac{t^3}{3!}(\beta - 6\beta^2 - 10\beta^3 + 5\beta^4) \\ &+ \frac{t^4}{3}(-1+\beta)^2\beta^2(-1+2\beta) \\ &- \frac{t^5}{60}(-1+\beta)^2\beta^2(3-20\beta+20\beta^2) \\ &+ \frac{t^6}{18}(-1+\beta)^3\beta^3(-1+2\beta) - \frac{t^7}{63}(-1+\beta)^4\beta^4, \\ &\vdots \end{aligned}$$

gives the solution in a closed form:

$$u(x,t) = \frac{\beta \exp t}{1-\beta+\beta\exp t}.$$

Table 3 shows the numerical results.

Table 3. Numerical results for Fisher's equation.

t	$\beta = 0.2$		$\beta = 0.8$	
0	0	0	0	0
0.2	6.19201E-06	1.96407E-06	5.57339E-06	2.63705E-06
0.4	1.03635E-04	2.60211E-05	1.13137E-04	4.7283E-05
0.6	5.45505E-04	1.05333E-04	4.00147E-04	2.63379E-04
0.8	1.78050E-03	2.54998E-04	1.18584E-03	9.01304E-04
1	4.45699E-03	4.515414E-04	1.99502E-03	1.60455E-05

3.4. Example 4

Consider the Fisher's equation of the form

$$u_t - u_{xx} - 6u(1-u) = 0,$$

subject to the initial conditions

$$u(x,0) = \frac{1}{(1+\mathrm{e}^x)^2}.$$

This Fisher's problem can be formulated as the integral equation

$$u(x,t) = \frac{1}{(1+\mathrm{e}^x)^2} + \int_0^t (u_{xx} + 6u(1-u))\mathrm{d}t.$$

Applying the convex homotopy method

$$\begin{aligned} &u_0 + pu_1 + p^2u_2 + \ldots = \\ &\frac{1}{(1+\mathrm{e}^x)^2} + p\int_0^t \left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \ldots\right) \\ &+ 6(u_0 + pu_1 + p^2u_2 + \ldots) \\ &\quad \cdot (1-(u_0 + pu_1 + p^2u_2 + \ldots))\mathrm{d}t, \end{aligned}$$

Table 4. Numerical results for Fisher's equation.

x	$t=0.2$		$t=0.4$	
0	7.22002E-03	2.69962E-03	5.75298E-02	5.01844E-02
0.2	9.89049E-03	2.15566E-03	1.6115E-01	5.27127E-02
0.4	1.09765E-02	1.136097E-03	1.39113E-01	4.12072E-02
0.6	1.04039E-02	7.38299E-04	1.51579E-01	2.25459E-02
0.8	8.50732E-03	5.73101E-04	1.43529E-01	5.28693E-03
1	5.87222E-03	9.07727E-04	1.19333E-01	4.23672E-03

and comparing the coefficients of equal powers of p

$$p^{(0)}: u_0(x,t)=\frac{1}{(1+e^x)^2},$$

$$p^{(1)}: u_1(x,t)=\frac{10}{(1+e^x)^2}t,$$

$$p^{(2)}: u_2(x,t)=\frac{1}{(1+e^x)^6}\Big[25e^x(1+e^x)^2(-1+2e^x)t^2 -200e^{2x}t^3+\frac{1}{(1+e^x)^3}((1+e^x)^6-1) \cdot(1+e^x(1+10t))\Big],$$

$$\begin{aligned}p^{(3)}: u_3(x,y)=&\frac{25e^x}{3(1+e^x)^6}\big((5-6e^x-15e^{2x} \\ &+20e^{3x})t^3\big)-\frac{50e^{2x}}{(1-e^x)^8}\left(-17+5e^x+52e^{2x}t^4\right) \\ &+\frac{150e^{2x}}{(1+e^x)^9}\left(5-47e^x+20e^{3x}t^5\right) \\ &+\frac{10000e^{3x}(-1+2e^x)t^6}{(1+e^x)^{10}}-\frac{240000e^{4x}t^7}{7(1+e^x)^{12}} \\ &+\frac{1}{(1+e^x)^6}\left[25e^x\left(1+e^x(1+e^x)^2\right)(-1+2e^x)t^2\right. \\ &\left.-200e^{2x}t^3+(1+e^x(1+10t))\right], \\ &\vdots\end{aligned}$$

gives the solution in a closed form:

$$u(x,t)=\frac{1}{(1+\exp(x-5t))^2}.$$

Table 4 shows the numerical results.

3.5. Example 5

Consider the generalized Fisher's equation

$$u_t=u_{xx}+u(1-u^6),$$

subject to the initial conditions

$$u(x,0)=\frac{1}{\left(1+e^{\frac{3}{2}x}\right)^{\frac{1}{3}}}.$$

This Fisher's problem can be formulated as the integral equation

$$u(x,t)=\frac{1}{\left(1+e^{\frac{3}{2}x}\right)^{\frac{1}{3}}}-p\int_0^t(u_{xx}+u(1-u^6))\mathrm{d}t.$$

Applying the convex homotopy method

$$\begin{aligned}u_0+pu_1+p^2u_2+\ldots=&\frac{1}{\left(1+e^{\frac{3x}{2}}\right)^{1/3}} \\ &-p\int_0^t\left(\frac{\partial^2u_0}{\partial x^2}+p\frac{\partial^2u_1}{\partial x^2}+p^2\frac{\partial^2u_2}{\partial x^2}+\ldots\right) \\ &+(u_0+pu_1+p^2u_2+\ldots) \\ &\cdot(1-(u_0+pu_1+p^2u_2+\ldots)^6)\mathrm{d}t,\end{aligned}$$

and comparing the coefficients of equal powers of p

$$p^{(0)}: u_0(x,t)=\frac{1}{\left(1+e^{\frac{3}{2}x}\right)^{1/3}},$$

$$p^{(1)}: u_1(x,t)=\frac{4+e^{3x}(4+3t)+e^{3x/2}(8+11t)}{4(1+e^{3x/2})^{7/3}},$$

$$\begin{aligned}p^{(2)}: u_2(x,t)=&\frac{1}{131072(1+e^{3x/2})^{49/3}}\Big[4096e^{3x/2} \\ &\cdot(1+e^{3x/2})^{12}\Big(-363+403e^{3x/2}-9e^{3x}+9e^{9x/2}\Big)t^2 \\ &-57344e^{3x}(1+e^{3x/2})^{10}(11+3e^{3x/2})^2t^3 \\ &-17920e^{9x/2}(1+e^{3x/2})^8(11+3e^{3x/2})^3t^4 \\ &-3584e^{6x}(1+e^{3x/2})^6(11+3e^{3x/2})^4t^5 \\ &-448e^{15x/2}(1+e^{3x/2})^4(11+3e^{3x/2})^5t^6 \\ &-448e^{15x/2}(1+e^{3x/2})^4(11+3e^{3x/2})^5t^6 \\ &-32e^{9x}(1+e^{3x/2})^2(11+3e^{3x/2})^6t^7 \\ &-e^{21x/2}(11+3e^{3x/2})^7t^8+32768(1+e^{3x/2})^{14} \\ &\cdot(4+e^{3x}(4+3t)+e^{3x/2}(8+11t))\Big], \\ &\vdots\end{aligned}$$

Table 5. Numerical results for the generalized Fisher's equation.

x	$t=0.2$		$t=0.4$	
0	5.24926E-02	4.54137E-02	1.21845E-01	1.97465E-01
0.2	7.79547E-02	4.1746E-02	2.17494E-01	8.39974E-02
0.4	1.10805E-01	3.23276E-02	3.4171E-01	9.22231E-04
0.6	1.51375E-01	1.91936E-02	4.94354E-01	4.10631E-02
0.8	1.99601E-01	5.03821E-03	6.74017E-01	4.10631E-02
1	2.55137E-01	7.85833E-03	8.78892E-01	1.46625E-02

gives the solution in a closed form:

$$u(x,t) = ((1/2)\tanh[-3/4(x-5/2t)]+1/2)^{1/3}$$

Table 5 shows the numerical results.

3.6. Example 6

Consider the singularly perturbed sixth-order Boussinesq equation [14, 15, 32, 33, 40]

$$u_{tt} = u_{xx} + (p(u))_{xx} + \alpha u_{xxxx} + \beta u_{xxxxxx}.$$

Taking $\alpha = 1$, $\beta = 0$, and $p(u) = 3u^2$, the model equation is given as

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx}$$

with the initial conditions

$$u(x,0) = \frac{2ak^2 e^{kx}}{(1+ae^{kx})^2},$$
$$u_t(x,0) = \frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3},$$

where a and k are arbitrary constants. The exact solution $u(x,t)$ of the problem is given as [33]

$$u(x,t) = 2\frac{ak^2\exp(kx+k\sqrt{1+k^2}t)}{(1+a\exp(kx+k\sqrt{1+k^2}t))^2}.$$

Applying the convex homotopy method

$$\begin{aligned} &u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots = \\ &\frac{2ak^2e^{kx}}{(1+ae^{kx})^2} + \frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t \\ &+p\int_0^t\int_0^t\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \ldots\right)\mathrm{d}t\,\mathrm{d}t \\ &+p\int_0^t\int_0^t\Bigg[\left(\frac{\partial^4 u_0}{\partial x^4} + p\frac{\partial^4 u_1}{\partial x^4} + p^2\frac{\partial^4 u_2}{\partial x^4} + \ldots\right) \\ &+3\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \ldots\right)^2\Bigg]\mathrm{d}t\,\mathrm{d}t, \end{aligned}$$

and comparing the coefficients of equal powers of p

$$\begin{aligned} p^{(0)} &: u_0(x,t) = \frac{2e^{kx}}{(1+e^x)^2}, \\ p^{(1)} &: u_1(x,t) = \frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t \\ &\quad + \frac{2e^x(1-4e^x+e^{2x})}{(1+e^x)^4}t^2, \end{aligned}$$

$$\begin{aligned} p^{(2)} &: u_2(x,t) = -\frac{2\sqrt{2}e^x(-1+e^x)(1-10e^x+e^{2x})}{3(1+e^x)^5}t^3 \\ &+\frac{e^x(1-4e^x+e^{2x})(1-44e^x+78e^{2x}-44e^{3x}+e^{4x})}{3(1+e^x)^8}t^4, \end{aligned}$$

$$\begin{aligned} p^{(3)} &: u_3(x,t) = -\frac{1}{15(1+e^x)^7}\left[\sqrt{2}e^x(-1+e^x)\right. \\ &\quad \left.\cdot(1-56e^x+246e^{2x}-56e^{3x}+e^{4x})\right]t^5 \\ &\quad + \frac{1}{45(1+e^x)^{12}}\left[e^x(1-452e^x+19149e^{2x}\right. \\ &\quad \left.-207936e^{3x}+807378e^{4x}-1256568e^{5x})\right]t^6 \\ &\quad + \frac{1}{45(1+e^x)^{12}}\Big[e^x(807378e^{6x}-207936e^{7x} \\ &\quad +19149e^{8x}-452e^{9x}+e^{10x})\Big]t^6, \\ &\vdots \end{aligned}$$

gives the series solution as

$$\begin{aligned} u(x,t) &= \frac{2e^{kx}}{(1+e^x)^2} + \frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t \\ &+\frac{2e^x(1-4e^x+e^{2x})}{(1+e^x)^4}t^2 \\ &-\frac{2\sqrt{2}e^x(-1+e^x)(1-10e^x+e^{2x})}{3(1+e^x)^5}t^3 \\ &+\frac{e^x(1-4e^x+e^{2x})(1-44e^x+78e^{2x}-44e^{3x}+e^{4x})}{3(1+e^x)^8}t^4 \\ &+\frac{8e^{2x}(1-10e^x+20e^{2x}-10e^{3x}+e^{4x})}{(1+e^x)^8}t^4 \\ &-\frac{\sqrt{2}e^x(-1+e^x)(1-56e^x+246e^{2x}-56e^{3x}+e^{4x})}{15(1+e^x)^7}t^5 \\ &+\frac{1}{45(1+e^x)^{12}}\left[e^x(1-452e^x+19149e^{2x}\right. \\ &\quad \left.-207936e^{3x}+807378e^{4x}-1256568e^{5x})\right]t^6 \\ &+\frac{1}{45(1+e^x)^{12}}\left[e^x(807378e^{6x}-207936e^{7x}\right. \\ &\quad \left.+19149e^{8x}-452e^{9x}+e^{10x})\right]t^6 + \ldots. \end{aligned}$$

Table 6. Error estimates.

x_i	t_j 0.01	0.02	0.04	0.1	0.2	0.5
−1	2.80886 E-14	1.79667 E-12	1.15235 E-10	2.83355 E-8	1.83899 E-6	4.74681 E-4
−0.8	6.27276 E-14	4.01362 E-12	2.57471 E-10	6.33178 E-8	4.10454 E-6	1.04489 E-3
−0.6	6.08402 E-14	3.90188 E-12	2.25663 E-10	6.18024 E-8	4.02299 E-6	1.03093 E-3
−0.4	1.16573 E-14	7.41129 E-13	4.82756 E-11	1.23843 E-8	8.53800 E-6	2.46302 E-4
−0.2	5.53446 E-14	3.53395 E-12	2.25663 E-10	5.47485 E-8	3.47264 E-6	8.35783 E-4
0	8.63198 E-14	5.53357 E-12	2.54174 E-10	8.65197 E-8	5.54893 E-6	1.37353 E-3
0.2	5.56222 E-14	3.55044 E-12	2.27779 E-10	5.60362 E-8	3.63600 E-6	9.29612 E-4
0.4	1.14353 E-14	7.14928 E-13	4.49107 E-11	1.03370 E-8	5.93842 E-7	9.61260 E-5
0.6	6.06182 E-14	3.87551 E-12	2.47218 E-10	5.97562 E-8	3.76275 E-6	8.79002 E-4
0.8	6.23945 E-14	3.99519 E-12	2.55127 E-10	6.18881 E-8	3.92220 E-6	9.36404 E-4
1	2.79776 E-14	1.78946 E-12	1.14307 E-10	2.77684 E-8	1.76607 E-6	4.28986 E-4

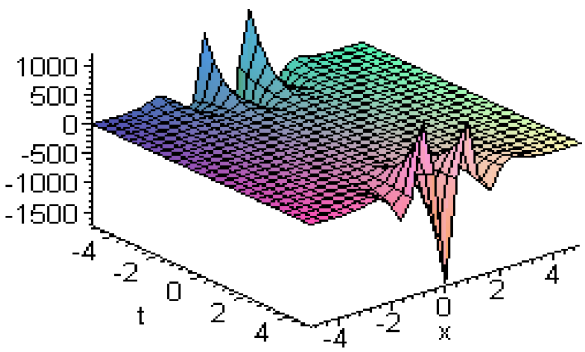

Fig. 1. Series solution $u(x,t)$.

Table 6 exhibits the absolute error between the exact and the series solutions. Higher accuracy can be obtained by introducing some more components of the series solution. Figure 1 depicts the series solution $u(x,t)$.

3.7. Example 7

Consider the singularly perturbed sixth-order Boussinesq equation [14, 15, 32, 33, 40]

$$u_{tt} = u_{xx} + (u^2)_{xx} - u_{xxxx} + \frac{1}{2}u_{xxxxxx}$$

with the initial conditions

$$u(x,0) = -\frac{105}{169}\operatorname{sech}^4\left(\frac{x}{\sqrt{26}}\right),$$

$$u_t(x,0) = \frac{-210\sqrt{\frac{194}{13}}\operatorname{sech}^4\left(\frac{x}{\sqrt{26}}\right)\tanh\left(\frac{x}{\sqrt{26}}\right)}{2197}.$$

The exact solution of the problem is given as

$$u(x,t) = -\frac{105}{169}\operatorname{sech}^4\left[\sqrt{\frac{1}{26}}\left(x - \sqrt{\frac{97}{169}}t\right)\right].$$

Applying the convex homotopy method

$$u_0 + pu_1 + p^2u_2 + \ldots = -\frac{105}{169}\operatorname{sech}^4\left(\frac{x}{\sqrt{26}}\right)$$
$$+\frac{-210\sqrt{\frac{194}{13}}\operatorname{sech}^4\left(\frac{x}{\sqrt{26}}\right)\tanh\left(\frac{x}{\sqrt{26}}\right)}{2197}t$$
$$+p\int_0^t\int_0^t\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \ldots\right)\mathrm{d}t\mathrm{d}t$$
$$+p\int_0^t\int_0^t\left[(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)^2_{xx}\right.$$
$$\left.-\left(\frac{\partial^4 u_0}{\partial x^4} + p\frac{\partial^4 u_1}{\partial x^4} + p^2\frac{\partial^4 u_2}{\partial x^4} + \ldots\right)\right]\mathrm{d}t\mathrm{d}t$$
$$+\frac{1}{2}p\int_0^t\int_0^t\left(\frac{\partial^6 u_0}{\partial x^6} + p\frac{\partial^6 u_1}{\partial x^6} + p^2\frac{\partial^6 u_2}{\partial x^6} + \ldots\right)\mathrm{d}t\mathrm{d}t,$$

and comparing the coefficients of equal powers of p

$$p^{(0)}: u_0 = -\frac{105}{169}\operatorname{sech}^4\left(\frac{x}{\sqrt{26}}\right),$$

$$p^{(1)}: u_1(x,t) = -\frac{105\sqrt{\frac{194}{13}}\operatorname{sech}^6\left(\frac{x}{\sqrt{26}}\right)\sinh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)}{2197}t$$
$$-\frac{105}{371293}\left(-291 + 194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right)\operatorname{sech}^6\frac{x}{\sqrt{26}}t^2,$$

$$p^{(2)}: u_2(x,t) = \frac{395\operatorname{sech}^7\frac{x}{\sqrt{26}}}{52206766144}\left(10816\sqrt{2522}\sinh\frac{x}{\sqrt{26}}\right.$$
$$\left.-1664\sqrt{2522}\sinh\frac{3x}{\sqrt{26}}\right)t^3 + \left[-334200\operatorname{sech}^5\left(\frac{x}{\sqrt{26}}\right)\right.$$
$$+354247\cosh\left(\frac{2}{\sqrt{13}}x\right)\operatorname{sech}^5\left(\frac{x}{\sqrt{26}}\right)$$

Table 7. Error estimates.

x_i	t_j 0.01	0.02	0.04	0.1	0.2	0.5
−1	7.77156 E-16	1.36557 E-14	8.57869 E-13	2.09264 E-10	1.33823 E-8	3.25944 E-6
−0.8	1.11022 E-16	1.99840 E-15	1.12688 E-13	2.73880 E-11	1.74288 E-9	4.14094 E-7
−0.6	2.22045 E-16	1.09912 E-14	7.28861 E-13	1.78030 E-10	1.14025 E-8	2.79028 E-6
−0.4	1.11022 E-16	2.32037 E-14	1.50302 E-12	3.67002 E-10	2.34944 E-8	5.74091 E-6
−0.2	6.66134 E-16	3.23075 E-14	2.04747 E-12	4.99918 E-10	3.19983 E-9	7.81509 E-6
0	4.44089 E-16	3.49720 E-14	2.24365 E-12	5.47741 E-10	3.50559 E-8	8.55935 E-6
0.2	5.55112 E-16	3.19744 E-14	2.04714 E-12	4.99820 E-10	3.19858 E-8	7.80749 E-6
0.4	3.33067 E-16	2.32037 E-14	1.50324 E-12	3.66815 E-10	2.34706 E-8	5.72641 E-6
0.6	3.33067 E-16	1.12133 E-14	7.28528 E-12	1.77772 E-10	1.13695 E-8	2.77022 E-6
0.8	3.33067 E-16	1.99840 E-15	1.13132 E-13	2.76944 E-11	1.78208 E-9	4.41936 E-7
1	7.77156 E-16	1.38778 E-14	8.58313 E-13	2.09593 E-10	1.34244 E-8	3.28504 E-6

$$
\begin{aligned}
&-47164\cosh\left(\frac{2\sqrt{2}}{\sqrt{13}}x\right)\mathrm{sech}^5\left(\frac{x}{\sqrt{26}}\right)\Bigg]t^4\\
&+\Bigg[3201\cosh^3\left(\frac{3\sqrt{2}}{\sqrt{13}}x\right)\mathrm{sech}^5\left(\frac{x}{\sqrt{26}}\right)\\
&-388\cosh\left(\frac{4\sqrt{2}}{\sqrt{13}}x\right)\mathrm{sech}^5\left(\frac{x}{\sqrt{26}}\right)\Bigg]t^4,\\
&\vdots
\end{aligned}
$$

gives the series solution as

$$
\begin{aligned}
u(x,t) = &-\frac{105}{169}\mathrm{sech}^4\left(\frac{x}{\sqrt{26}}\right)\\
&-\frac{105\sqrt{\frac{194}{13}}\mathrm{sech}^6\left(\frac{x}{\sqrt{26}}\right)\sinh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)}{2197}t\\
&-\frac{105}{371293}\left(-291+194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right)\mathrm{sech}^6\frac{x}{\sqrt{26}}t^2\\
&+\frac{395\mathrm{sech}^7\frac{x}{\sqrt{26}}}{52206766144}\Bigg(10816\sqrt{2522}\sinh\frac{x}{\sqrt{26}}\\
&-1664\sqrt{2522}\sinh\frac{3x}{\sqrt{26}}\Bigg)t^3+\Bigg[-334200\mathrm{sech}^5\left(\frac{x}{\sqrt{26}}\right)\\
&+354247\cosh\left(\frac{2}{\sqrt{13}}x\right)\mathrm{sech}^5\left(\frac{x}{\sqrt{26}}\right)\\
&-47164\cosh\left(\frac{2\sqrt{2}}{\sqrt{13}}x\right)\mathrm{sech}^5\left(\frac{x}{\sqrt{26}}\right)\Bigg]t^4\\
&+\Bigg[3201\cosh^3\left(\frac{3\sqrt{2}}{\sqrt{13}}x\right)\mathrm{sech}^5\left(\frac{x}{\sqrt{26}}\right)\\
&-388\cosh\left(\frac{4\sqrt{2}}{\sqrt{13}}x\right)\mathrm{sech}^5\left(\frac{x}{\sqrt{26}}\right)\Bigg]t^4+\dots.
\end{aligned}
$$

Table 7 exhibits the absolute error between the exact and the series solutions. Higher accuracy can be obtained by introducing some more components of the series solution. Figure 2 depicts the series solution $u(x,t)$.

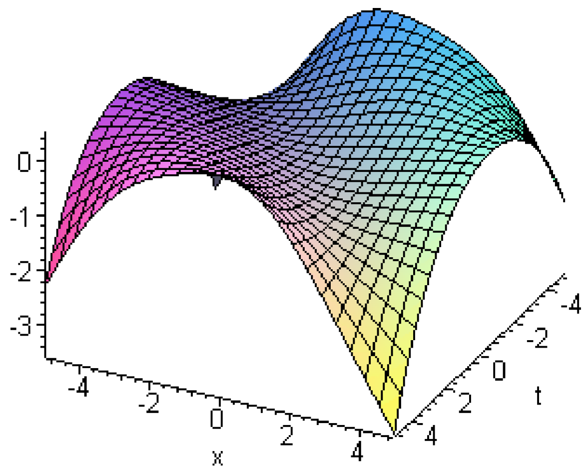

Fig. 2. Series solution $u(x,t)$.

3.8. Example 8

Consider the following nonlinear system of partial differential equations:

$$
\begin{aligned}
&u_t+v_xw_y-v_yw_x=-u, \quad v_t+w_xu_y+w_yu_x=v,\\
&w_t+u_xv_y+u_yv_x=-w,
\end{aligned}
$$

with the initial conditions

$$
u(x,y,0)=\mathrm{e}^{x+y}, \; v(x,y,0)=\mathrm{e}^{x-y}, \; w(x,y,0)=\mathrm{e}^{-x+y}.
$$

Applying the convex homotopy method

$$
\begin{aligned}
&u_0+pu_1+p^2u_2+\dots=\mathrm{e}^{x+y}-p\int_0^t\left(\frac{\partial v_0}{\partial x}+p\frac{\partial v_1}{\partial x}\right.\\
&\left.+p^2\frac{\partial v_2}{\partial x}+\dots\right)\left(\frac{\partial w_0}{\partial y}+p\frac{\partial w_1}{\partial y}+p^2\frac{\partial w_2}{\partial y}+\dots\right)\mathrm{d}t\\
&+p\int_0^t\left(\frac{\partial v_0}{\partial y}+p\frac{\partial v_1}{\partial y}+p^2\frac{\partial v_2}{\partial y}+\dots\right)\left(\frac{\partial w_0}{\partial x}+p\frac{\partial w_1}{\partial x}\right.\\
&\left.+p^2\frac{\partial w_2}{\partial x}+\dots\right)\mathrm{d}t-p_0^t(u_0+pu_1+p^2u_2+\dots)\mathrm{d}t,
\end{aligned}
$$

$$v_0+pv_1+p^2v_2+\ldots=\mathrm{e}^{x-y}-p\int_0^t\left(\frac{\partial u_0}{\partial y}+p\frac{\partial u_1}{\partial y}+p^2\frac{\partial u_2}{\partial y}+\ldots\right)\left(\frac{\partial w_0}{\partial x}+p\frac{\partial w_1}{\partial x}+p^2\frac{\partial w_2}{\partial x}+\ldots\right)\mathrm{d}t$$
$$-p\int_0^t\left(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+\ldots\right)\left(\frac{\partial w_0}{\partial y}+p\frac{\partial w_1}{\partial y}+p^2\frac{\partial w_2}{\partial y}+\ldots\right)\mathrm{d}t+p\int_0^t\left(v_0+pv_1+p^2v_2+\ldots\right)\mathrm{d}t,$$

$$w_0+pw_1+p^2w_2+\ldots=\mathrm{e}^{-x+y}-p\int_0^t\left(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+\ldots\right)\left(\frac{\partial v_0}{\partial y}+p\frac{\partial v_1}{\partial y}+p^2\frac{\partial v_2}{\partial y}+\ldots\right)\mathrm{d}t$$
$$-p\int_0^t\left(\frac{\partial u_0}{\partial y}+p\frac{\partial u_1}{\partial y}+p^2\frac{\partial u_2}{\partial y}+\ldots\right)\left(\frac{\partial v_0}{\partial x}+p\frac{\partial v_1}{\partial x}+p^2\frac{\partial v_2}{\partial x}+\ldots\right)\mathrm{d}t-p\int_0^t\left(w_0+pw_1+p^2w_2+\ldots\right)\mathrm{d}t,$$

and comparing the coefficients of equal powers of p

$$p^{(0)}:u_0(x,y,t)=\mathrm{e}^{x+y},\quad v_0(x,y,t)=\mathrm{e}^{x-y},\quad w_0(x,y,t)=\mathrm{e}^{-x+y},$$

$$p^{(1)}:u_1(x,y,t)=-t\mathrm{e}^{x+y},\quad v_1(x,y,t)=t\mathrm{e}^{x-y},\quad w_1(x,y,t)=t\mathrm{e}^{-x+y},$$

$$p^{(2)}:u_2(x,y,t)=\frac{t^2}{2!}\mathrm{e}^{x+y},\quad v_2(x,y,t)=\frac{t^2}{2!}\mathrm{e}^{x-y},\quad w_2(x,y,t)=\frac{t^2}{2!}\mathrm{e}^{-x+y},$$

$$p^{(3)}:u_3(x,y,t)=\frac{t^3}{3!}\mathrm{e}^{x+y},\quad v_3(x,y,t)=\frac{t^3}{3!}\mathrm{e}^{x-y},\quad w_3(x,y,t)=\frac{t^3}{3!}\mathrm{e}^{-x+y},$$
$$\vdots$$

gives the closed form solution as

$$(u,v,w)=(\mathrm{e}^{x+y-t},\mathrm{e}^{x-y+t},\mathrm{e}^{-x+y+t}).$$

3.9. Example 9

Consider the singular fourth-order parabolic equation

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\left(\frac{\partial^4 u}{\partial x^4}\right)=0,\quad \frac{1}{2}<x<1,\quad t>0,$$

subject to the initial conditions

$$u(x,0)=0,\quad \frac{1}{2}<x<1,$$
$$\frac{\partial u}{\partial t}(x,0)=1+\frac{x^5}{120},\quad \frac{1}{2}<x<1,$$

and the boundary conditions

$$u\left(\frac{1}{2},t\right)=\left(1+\frac{(1/2)^5}{120}\right)\sin t,$$
$$u(1,t)=\left(\frac{121}{120}\right)\sin t,\quad t>0,$$
$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},t\right)=\frac{1}{6}\left(\frac{1}{2}\right)^3\sin t,$$
$$\frac{\partial^2 u}{\partial x^2}(1,t)=\frac{1}{6}\sin t,\quad t>0.$$

Applying the convex homotopy method

$$u_0+pu_1+p^2u_2+\ldots=u_0(x,t)-p\int_0^t\int_0^t\left(\frac{1}{x}+\frac{x^4}{120}\right)\cdot\left(\frac{\partial^4 u_0}{\partial x^4}+p\frac{\partial^4 u_1}{\partial x^4}+p^2\frac{\partial^4 u_2}{\partial x^4}+\ldots\right)\mathrm{d}t\mathrm{d}t,$$

and comparing the coefficients of equal powers of p

$$p^{(0)}:u_0(x,t)=\left(1+\frac{x^5}{120}\right)t,$$
$$p^{(1)}:u_1(x,t)=-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!},$$
$$p^{(2)}:u_2(x,t)=\left(1+\frac{x^5}{120}\right)\frac{t^5}{5!},$$
$$p^{(3)}:u_3(x,t)=-\left(1+\frac{x^5}{120}\right)\frac{t^7}{7!},$$
$$\vdots$$

gives the solution as

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}\ldots\right)$$
$$= \left(1+\frac{x^5}{120}\right)\sin t,$$

which is the exact solution. It is interesting to point out that the exact solution is obtained by using the initial conditions only. Moreover, the obtained solution can be used to justify the given boundary conditions.

3.10. Example 10

Consider the following singular fourth-order parabolic partial differential equation in two space variables:

$$\frac{\partial^2 u}{\partial t^2}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}=0$$

with the initial conditions

$$u(x,y,0)=0, \quad \frac{\partial u}{\partial t}(x,y,0)=2+\frac{x^6}{6!}+\frac{y^6}{6!},$$

and the boundary conditions

$$u\left(\frac{1}{2},y,t\right)=\left(2+\frac{(0.5)^6}{6!}+\frac{y^6}{6!}\right)\sin t,$$

$$u(1,y,t)=\left(2+\frac{1}{6!}+\frac{y^6}{6!}\right)\sin t,$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},y,t\right)=\frac{(0.5)^4}{24}\sin t,$$

$$\frac{\partial^2 u}{\partial x^2}(1,y,t)=\frac{1}{24}\sin t,$$

$$\frac{\partial^2 u}{\partial y^2}\left(x,\frac{1}{2},t\right)=\frac{(0.5)^4}{24}\sin t,$$

$$\frac{\partial^2 u}{\partial y^2}(x,1,t)=\frac{1}{24}\sin t.$$

Applying the convex homotopy method

$$u_0+pu_1+p^2u_2+\ldots=u_0(x,t)$$
$$-2p\int_0^t\int_0^t\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\left(\frac{\partial^4 u_0}{\partial y^4}+p\frac{\partial^4 u_1}{\partial y^4}+p^2\frac{\partial^4 u_2}{\partial y^4}+\ldots\right)\mathrm{d}t\mathrm{d}t$$
$$-2p\int_0^t\int_0^t 2\left(\frac{1}{x^2}+\frac{x^4}{120}\right)\left(\frac{\partial^4 u_0}{\partial x^4}+p\frac{\partial^4 u_1}{\partial x^4}+p^2\frac{\partial^4 u_2}{\partial x^4}+\ldots\right)\mathrm{d}t\mathrm{d}t,$$

and comparing the coefficients of equal powers of p

$$p^{(0)}: u_0(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t,$$

$$p^{(1)}: u_1(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(-\frac{t^3}{3!}\right),$$

$$p^{(2)}: u_2(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(\frac{t^5}{5!}\right),$$

$$p^{(3)}: u_3(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(-\frac{t^7}{7!}\right),$$

$$p^{(4)}: u_4(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(\frac{t^9}{9!}\right),$$

$$\vdots$$

gives the exact solution easily:

$$u(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\frac{t^9}{9!}+\ldots\right)$$
$$=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin t.$$

3.11. Example 11

Consider the fourth-order singular parabolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 u}{\partial x^4}=0, \quad 0<x<1, \quad t>0$$

with the initial conditions

$$u(x,0)=x-\sin x, \quad 0<x<1,$$
$$\frac{\partial u}{\partial t}(x,0)=-(x-\sin x), \quad 0<x<1,$$

and the boundary conditions

$$u(0,t)=0,\quad u(1,t)=\mathrm{e}^{-t}(1-\sin 1),\quad t>0,$$
$$\frac{\partial^2 u}{\partial x^2}(0,t)=0,\quad \frac{\partial^2 u}{\partial x^2}(1,t)=\mathrm{e}^{-t}\sin 1,\quad t>1.$$

Applying the convex homotopy method

$$u_0+pu_1+p^2u_2+\ldots=u_0(x,t)-p\int_0^t\int_0^t\left(\frac{x}{\sin x}-1\right)\cdot\left(\frac{\partial^4 u_0}{\partial x^4}+p\frac{\partial^4 u_1}{\partial x^4}+p^2\frac{\partial^4 u_2}{\partial x^4}+\ldots\right)\mathrm{d}t\mathrm{d}t,$$

and comparing the coefficients of equal powers of p

$$p^{(0)}: u_0(x,t)=x-\sin x,$$
$$p^{(1)}: u_1(x,t)=-(x-\sin x)t,$$
$$p^{(2)}: u_2(x,t)=(x-\sin x)\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right),$$
$$p^{(3)}: u_3(x,t)=(x-\sin x)\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right),$$
$$p^{(4)}: u_4(x,t)=(x-\sin x)\left(\frac{t^6}{6!}-\frac{t^7}{7!}\right),$$
$$p^{(5)}: u_5(x,t)=(x-\sin x)\left(\frac{t^8}{8!}-\frac{t^9}{9!}\right),$$
$$\vdots$$

gives the solution as

$$u(x,t)=(x-\sin x)\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\ldots\right)=(x-\sin x)\mathrm{e}^{-t},$$

which is the exact solution. It satisfies the boundary conditions also that we did not use in the analysis.

3.12. Example 12

Consider the two-dimensional initial boundary value problem

$$u_{tt}=\frac{1}{2}y^2u_{xx}+\frac{1}{2}x^2u_{yy},\quad 0<x,\quad y<1,\quad t>0$$

with the boundary conditions

$$u(0,y,t)=y^2\mathrm{e}^{-t},\quad u(1,y,t)=(1+y^2)\mathrm{e}^{-t},$$
$$u(x,0,t)=y^2\mathrm{e}^{-t},\quad u(x,1,t)=(1+x^2)\mathrm{e}^{-t},$$

and the initial conditions

$$u(x,y,0)=x^2+y^2,\quad u_t(x,y,0)=-(x^2+y^2).$$

Applying the convex homotopy method

$$u_0+pu_1+p^2u_2+\ldots=(x^2+y^2)+(x^2+y^2)t$$
$$+\frac{1}{2}p\int_0^t\int_0^t y^2\left(\frac{\partial^2 u_0}{\partial x^2}+p\frac{\partial^2 u_1}{\partial x^2}+p\frac{\partial^2 u_1}{\partial x^2}+\ldots\right)^2\mathrm{d}t\mathrm{d}t$$
$$+\frac{1}{2}p\int_0^t\int_0^t x^2\left(\frac{\partial^2 u_0}{\partial y^2}+p\frac{\partial^2 u_1}{\partial y^2}+p\frac{\partial^2 u_1}{\partial y^2}+\ldots\right)^2\mathrm{d}t\mathrm{d}t,$$

and comparing the coefficients of equal powers of p

$$p^{(0)}: u_0(x,y,t)=(x^2+y^2)-(x^2+y^2)t,$$
$$p^{(1)}: u_1(x,y,t)=(x^2+y^2)\frac{t^2}{2!}-(x^2+y^2)\frac{t^3}{3!},$$
$$p^{(2)}: u_2(x,y,t)=(x^2+y^2)\frac{t^4}{4!}-(x^2+y^2)\frac{t^5}{5!},$$
$$p^{(3)}: u_3(x,y,t)=(x^2+y^2)\frac{t^5}{5!}-(x^2+y^2)\frac{t^7}{7!},$$
$$p^{(4)}: u_4(x,y,t)=(x^2+y^2)\frac{t^8}{8!}-(x^2+y^2)\frac{t^9}{9!},$$
$$p^{(5)}: u_5(x,y,t)=(x^2+y^2)\frac{t^{10}}{10!}-(x^2+y^2)\frac{t^{11}}{11!},$$
$$\vdots$$

gives the series solution as

$$u(x,y,t)=(x^2+y^2)\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\frac{t^6}{6!}-\frac{t^7}{7!}+\frac{t^8}{8!}-\ldots\right),$$

and, in a closed form, as

$$u(x,y,t)=(x^2+y^2)\mathrm{e}^{-t},$$

which is in full agreement with [12].

3.13. Example 13

Consider the three-dimensional initial boundary value problem

$$u_{tt}=\frac{1}{45}x^2u_{xx}+\frac{1}{45}y^2u_{yy}+\frac{1}{45}z^2u_{zz}-u,$$
$$0<x,\quad y<1,\quad t<0,$$

subject to the Neumann boundary conditions

$$u_x(0,y,z,t) = 0,$$
$$u_x(1,y,z,t) = 6y^6z^6\sinh t,$$
$$u_y(x,0,z,t) = 0,$$
$$u_y(x,1,z,t) = 6x^6z^6\sinh t,$$
$$u_z(x,y,0,t) = 0,$$
$$u_z(x,y,1,t) = 6x^6y^6\sinh t,$$

and the initial conditions

$$u(x,y,z,0) = 0, \quad u_t(x,y,z,0) = x^6y^6z^6.$$

Applying the convex homotopy method

$$\begin{aligned}
&u_0 + pu_1 + p^2u_2 + \ldots = (x^6y^6z^6)t \\
&+\frac{1}{45}p\int_0^t\int_0^t x^2\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p\frac{\partial^2 u_2}{\partial x^2} + \ldots\right)^2 \mathrm{d}t\mathrm{d}t \\
&+\frac{1}{45}p\int_0^t\int_0^t y^2\left(\frac{\partial^2 u_0}{\partial y^2} + p\frac{\partial^2 u_1}{\partial y^2} + p\frac{\partial^2 u_2}{\partial y^2} + \ldots\right)^2 \\
&+z^2\left(\frac{\partial^2 u_0}{\partial z^2} + p\frac{\partial^2 u_1}{\partial z^2} + p\frac{\partial^2 u_2}{\partial z^2} + \ldots\right)^2 \mathrm{d}t\mathrm{d}t \\
&-p\int_0^t\int_0^t (u_0 + pu_1 + p^2u_2 + \ldots)\mathrm{d}t\mathrm{d}t,
\end{aligned}$$

and comparing the coefficients of equal powers of p

$$p^{(0)} : u_0(x,y,z,t) = x^6y^6z^6t,$$
$$p^{(1)} : u_1(x,y,z,t) = x^6y^6z^6\frac{t^3}{3!},$$
$$p^{(2)} : u_2(x,y,z,t) = x^6y^6z^6\frac{t^5}{5!},$$
$$p^{(3)} : u_3(x,y,z,t) = x^6y^6z^6\frac{t^7}{7!},$$
$$p^{(4)} : u_4(x,y,z,t) = x^6y^6z^6\frac{t^9}{9!},$$
$$\vdots$$

gives the series solution as

$$\begin{aligned}
u(x,y,z,t) &= x^6y^6z^6\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{9!} + \ldots\right) \\
&= x^6y^6z^6\sinh t.
\end{aligned}$$

3.14. Example 14

Consider the two-dimensional nonlinear inhomogeneous initial boundary value problem

$$u_{tt} = 2x^2 + 2y^2 + \frac{15}{2}(xu_{xx}^2 + yu_{yy}^2),$$
$$0 < x, \quad y < 1, \quad t > 0$$

with the boundary conditions

$$u(0,y,t) = y^2t^2 + yt^6,$$
$$u(1,y,t) = (1+y^2)t^2 + (1+y)t^6,$$
$$u(x,0,t) = x^2t^2 + xt^6,$$
$$u(x,1,t) = (1+x^2)t^2 + (1+x)t^6,$$

and the initial conditions

$$u(x,y,0) = 0, \quad u_t(x,y,0) = 0.$$

Applying the convex homotopy method

$$\begin{aligned}
&u_0 + pu_1 + p^2u_2 + \ldots = p\int_0^t\int_0^t \left[x\left(\frac{\partial^2 \tilde{u}_0}{\partial x^2} + p\frac{\partial^2 \tilde{u}_1}{\partial x^2}\right.\right. \\
&\left.+p^2\frac{\partial^2 \tilde{u}_2}{\partial x^2} + \ldots\right)^2 + y\left(\frac{\partial^2 \tilde{u}_0}{\partial y^2} + p\frac{\partial^2 \tilde{u}_1}{\partial y^2}\right. \\
&\left.\left.+p^2\frac{\partial^2 \tilde{u}_2}{\partial y^2} + \ldots\right)^2 + (2x^2 + 2y^2)\right]\mathrm{d}t\mathrm{d}t,
\end{aligned}$$

and comparing the coefficients of equal powers of p

$$p^{(0)} : u_0(x,y,t) = 0,$$
$$p^{(1)} : u_1(x,y,t) = (x^2+y^2)t^2,$$
$$p^{(2)} : u_2(x,y,t) = (x+y)t^6,$$
$$p^{(3)} : u_3(x,y,t) = 0,$$
$$\vdots$$

gives the solution as

$$u(x,y,t) = (x^2+y^2)t^2 + (x+y)t^6,$$

which is in full agreement with [12].

3.15. Example 15

Consider the three-dimensional nonlinear initial boundary value problem

$$u_{tt} = (2-t^2) + u - (\mathrm{e}^{-x}u_{xx}^2 + \mathrm{e}^{-y}u_{yy}^2 + \mathrm{e}^{-z}u_{zz}^2),$$
$$0 < x, \quad y < 1, \quad t < 0,$$

subject to the Neumann boundary conditions

$$u_x(0,y,z,t)=1,\quad u_x(1,y,z,t)=\mathrm{e},$$

$$u_y(x,0,z,t)=0,\quad u_y(x,1,z,t)=\mathrm{e},$$

$$u_z(x,y,0,t)=1,\quad u_z(x,y,1,t)=\mathrm{e},$$

and the initial conditions

$$u(x,y,z,0)=\mathrm{e}^x+\mathrm{e}^y+\mathrm{e}^z,$$

$$u_t(x,y,z,0)=0.$$

Applying the convex homotopy method

$$u_0+pu_1+p^2u_2+\ldots=(\mathrm{e}^x+\mathrm{e}^y+\mathrm{e}^z)$$

$$-p\int_0^t\int_0^t\left[\mathrm{e}^{-x}\left(\frac{\partial^2u_0}{\partial x^2}+p\frac{\partial^2u_1}{\partial x^2}+p^2\frac{\partial^2u_2}{\partial x^2}+\ldots\right)^2\right.$$

$$\left.-(2-t^2)\right]\mathrm{d}t\mathrm{d}t-p\int_0^t\int_0^t\left[\mathrm{e}^{-y}\left(\frac{\partial^2u_0}{\partial y^2}+p\frac{\partial^2u_1}{\partial y^2}\right.\right.$$

$$\left.+p^2\frac{\partial^2u_2}{\partial y^2}+\ldots\right)^2+\mathrm{e}^{-z}\left(\frac{\partial^2u_0}{\partial z^2}+p\frac{\partial^2u_1}{\partial z^2}+p\frac{\partial u_2}{\partial z^2}\right.$$

$$\left.\left.+\ldots\right)^2\right]\mathrm{d}t\mathrm{d}t+p\int_0^t\int_0^t(u_0+pu_1+p^2u_2+\ldots)\mathrm{d}t\mathrm{d}t,$$

and comparing the coefficients of equal powers of p

$$p^{(0)}:u_0(x,y,z,t)=(\mathrm{e}^x+\mathrm{e}^y+\mathrm{e}^z)+t^2-\frac{t^4}{12},$$

$$p^{(1)}:u_1(x,y,z,t)=\frac{t^4}{12}-\frac{t^6}{360},$$

$$p^{(2)}:u_2(x,y,z,t)=\frac{t^6}{360}-\frac{t^8}{20160},$$

$$\vdots$$

gives the solution as

$$u(x,y,z,t)=(\mathrm{e}^x+\mathrm{e}^y+\mathrm{e}^z)+t^2.$$

4. Conclusions

We applied the homotopy perturbation method (HPM) for finding the solution of a system of partial differential equations. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. It may be concluded that the HPM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result. The fact that the HPM solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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